$$(x+1)^{p(p-1)} + (x+1)^{p(p-2)} + \cdots + (x+1)^p + 1 = 0$$

will take the form $$\chi^{p(p-1)} + p \cdot \chi(x),$$

if p is a prime number, and if " $\chi(x)$ is a polynomial with integral coefficients whose constant term is 1" (pages 22–23).

(4) To point out that although the *Conchoid of Nicomedes* is used in the text to trisect an angle, this application of the curve was the discovery of Pappus (about 300 A.D.), and *not* of Nicomedes (about 180 B.C.).[1] Nicomedes used the curve for the duplication of the cube.[2]

(5) To add the indication of proof that a prime number greater than any number we please *exists*. This is needed in the proof of the transcendence of e.

(6) To add, page 61, the proof that $\text{Lim}_{n \doteq \infty}\, x^n/\underline{|n} = 0$.

(7) To make the following corrections:

Page 12, *for* $F(x) = C\cdot[\phi(x)]^\nu$ *read* $F(x) = C_1\cdot[\phi(x)]^\nu$.

Page 14, line 8 should read

$$+ C_1 \sum_{\nu=1}^{\nu=N} \sum_r e^{|k_\nu|} c_r k_\nu^r q_{r,k_\nu} + C_2 \sum_{\nu=1}^{\nu=N'} e^{|l_\nu|} c_r' l_\nu^r q_{r l_\nu} + \cdots = 0.$$

Page 34, *for* "with the straight edge and one fixed circle we can solve every quadratic equation," *read* "with the straight edge and one fixed circle, the center being given, we can solve every quadratic equation for which line-segments corresponding to the coefficients are given."

Page 72, line 7, *for* b^{Np} *read* $b^{N'p}$; *for* l_N *read* $l_{N'}$.

Also in the theorem on page 5, some ambiguity would be avoided by setting $\phi(x) = 0$ for $f(x) = 0$.

Note. Since this article was written I have seen a new portion, Bd. III, Heft 5, of the *Encyklopädie der mathematischen Wissenschaften*, published at Leipzig on June 8, 1914. It contains a section by J. Sommer on "Elementare Geometrie vom Standpunkte der neueren Analysis aus," pages 773–858, and about half of the section, pages 859–962, by M. Zacharias on "Elementargeometrie und elementare nicht-Euklidische Geometrie in synthetischer Behandlung." Many parts of this Heft will be of interest in connection with questions discussed above. As to the Gaussian polygons, reference might have been given to paragraph 22 of L. O. Hölder's section of the *Encyklopädie*, published in 1899 and entitled, "Galois'sche Theorie mit Anwendungen."

ON THE TRISECTION OF AN ANGLE AND THE CONSTRUCTION OF REGULAR POLYGONS OF 7 AND 9 SIDES.

By L. E. DICKSON, University of Chicago.

1. Purpose and Plan of this Note. Frequently a wide-awake student who has learned how to bisect any angle asks if every angle can be trisected and, if not, why not; after learning how to construct regular polygons of 3, 4, 5, 6, 8

[1] Pappus, ed. Hultsch, p. 246.

[2] *Cf.* CANTOR, *Vorlesungen über Geschichte der Math.*, Bd. I, 3 Aufl., 1907, p. 351.

and 10 sides, he is apt to ask about the missing ones of 7 and 9 sides. Having several times received a first aid call from teachers of these inquisitive students, the writer would find it convenient to be able to refer to an exposition of these questions which is as elementary as possible. It is the purpose of this note to present such a treatment.

Moreover, it seems necessary that these questions be discussed publicly at regular intervals in order to keep down the number of angle-trisectors, who are partly unable and largely unwilling to understand the standard proofs of the impossibility of these constructions by means of ruler and compasses, but prefer to attempt to make the issue depend upon their own alleged construction involving always a confusing mass of lines and circles and always a child-like error.

With either class of readers, the use of imaginary numbers is not convincing. Hence they are not employed in this note, even though the imaginary roots of unity enter naturally into the questions concerning regular polygons. Moreover, the entire discussion is not beyond a college freshman.

2. The Cubic Equations. In the problem of the duplication of a cube, we take as the unit of length a side of the given cube, and seek the length x of a side of another cube whose volume is double that of the given cube; thus

$$x^3 = 2. \tag{1}$$

In the problem of the trisection of a given angle A, we are given a line of length $\cos A$ and seek a line of length $\cos (A/3)$. For, if we lay off the unit of length AB on one arm of angle A and draw the perpendicular BC to the other arm, the number of units of length in AC is $\cos A$ or $-\cos A$, according as A is an acute or obtuse angle. We employ the well-known trigonometric identity

$$\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3}.$$

Multiply each member by 2 and set $x = 2 \cos (A/3)$. Thus

$$x^3 - 3x = 2 \cos A.$$

We are to prove that an arbitrary[1] angle A cannot be trisected by ruler and compasses. It suffices to prove this for the angle $A=120°$. Then $\cos A=-1/2$ and the cubic is

$$x^3 - 3x + 1 = 0. \tag{2}$$

After we have proved that angle 120° cannot be so trisected and hence that angle 40° cannot be constructed by ruler and compasses, it will follow that a regular polygon of nine sides cannot be so constructed, since the angle at the center subtended by one side is $\frac{1}{9}\,360° = 40°$.

Finally, the problem of the construction of a regular polygon of seven sides by ruler and compasses is equivalent to the construction of angle B containing

[1] Certain angles, like $A = 180°$, can be trisected. Since $\cos 60° = 1/2$, the cubic then has the root $x = 1$. Hence this case does not invalidate our general theorem in § 4.

$\frac{360}{7}$ degrees and hence to the construction of a line of length $x = 2\cos B$. We have

$$\cos 3B = \cos(360^\circ - 3B) = \cos(7B - 3B) = \cos 4B,$$

$$\cos 3B = 4\cos^3 B - 3\cos B,$$

$$\cos 4B = 2\cos^2 2B - 1 = 2(2\cos^2 B - 1)^2 - 1.$$

After multiplication by 2 and setting $x = 2\cos B$, we get

$$x^3 - 3x = 4(\tfrac{1}{2}x^2 - 1)^2 - 2,$$

$$0 = x^4 - 4x^2 + 2 - x^3 + 3x = (x - 2)(x^3 + x^2 - 2x - 1).$$

But $x = 2$ would give $\cos B = 1$, whereas B is acute. Hence[1]

$$x^3 + x^2 - 2x - 1 = 0 \tag{3}$$

3. Our Cubic Equations Have No Rational Roots. Suppose for example that equation (2) has the root a/b, where a and b are integers with no common (integral) divisor greater than unity. Then

$$\frac{a^3}{b^3} - 3\frac{a}{b} + 1 = 0, \qquad \frac{a^3}{b} = 3ab - b^2 = \text{integer}.$$

Thus if $b \neq \pm 1$, b has a divisor greater than unity in common with a, contrary to hypothesis. Hence $b = \pm 1$ and the root is an integer.

If a root x of (2) is an integer, it divides x^3 and $3x$ and hence also the constant term 1, so that $x = \pm 1$. By trial, neither $+1$ nor -1 is a root. Hence (2) has no rational root.

The same discussion applies step by step to equation (3). In the case of (1), we must try also the divisors ± 2.

Hence each of our problems has led us to a cubic equation with rational coefficients having no rational root. Each problem is therefore impossible in view of the next theorem.

4. General Theorem. *It is not possible to construct by ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root.*

We begin by investigating the nature of a positive number p such that a line of length p can be constructed by ruler and compasses. The ends of this line as well as other points found in the course of the construction are located as the intersections of straight lines and circles. Consider the equations of these lines and circles referred to a fixed pair of rectangular axes, the y-axis not being parallel to any of our straight lines. The equation of any one of our lines is

$$y = mx + b. \tag{1}$$

[1] We can derive this equation and the corresponding ones for other regular polygons without the use of trigonometry, making use only of a theorem on chords in a circle. *Cf.* Dickson, *Annals of Mathematics*, 1894, p. 73.

Another line intersecting this has an equation

$$y = m'x + b'$$

and the coördinates of their point of intersection,

$$x = \frac{b' - b}{m - m'},\ y = \frac{mb' - m'b}{m - m'},$$

are rational functions of the coefficients of the lines.

To find the coördinates of the intersections of (1) with the circle

$$(x - e)^2 + (y - f\,)^2 = r^2,$$

we eliminate y and obtain a quadratic equation for x. Thus x, and hence also y, involves no irrationality (besides irrationalities already appearing in m, b, e, f, r) other than a square root.

Finally, the intersections of two circles are given by the intersections of one of them with their common chord, so that this case reduces to the preceding.

Hence the coördinates of the various points located by the construction, and therefore also the length p of the segment joining two of them, are found by a finite number of rational operations and extractions of real square roots, performed upon rational numbers or numbers obtained by such operations. By way of example, note that the side of a regular pentagon inscribed in a circle of radius unity is

$$\tfrac{1}{2}\sqrt{10 - 2\sqrt{5}}.$$

This point settled, consider a cubic equation with rational coefficients and having a constructible root r. Either r is rational or else it involves a real square root. In the latter case, we obtain a second root of the cubic by changing the sign of this square root in the expression for r. Then the third root of the cubic must be rational, since otherwise there would be, as before, a pair of roots in addition to the first pair. Hence in every case the cubic has a rational root, so that the denial of the general theorem stated at the beginning of this section leads to a contradiction. We have merely outlined in a rough way the final step of the proof. The argument in detail is accessible in books by Klein[1] and the writer;[2] it is based upon a systematic classification of the square roots involved in r, but employs only elementary algebraic principles.

The final step in the proof can be made in a few lines by means of the Galois theory of equations, which is based upon the theory of groups.

For a more elaborate elementary discussion of these special problems and the general problem relating to regular polygons, the reader may consult the eighth article in *Monographs on Modern Mathematics*, Longmans, Green and Co., 1911, where further references are given on page 386.

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[1] *Elementarmathematik vom höheren Standpunkte aus*, Leipzig, 1908, vol. 1, p. 125, and 2d ed., 1911.

[2] *Elementary Theory of Equations*, Wiley and Sons, 1914, p. 90.